Simple Walks Avoiding a Quadrant

Amélie Trotignon

Simon Fraser University SFU and Université François-Rabelais de Tours LMPT

Introduction

We consider the **simple walks** (*i.e.* walks with a set of steps $\mathcal{S}=\{\mathrm{W},\mathrm{N},\mathrm{E},\mathrm{S}\}$) in the lattice plane. We constrain the walks to avoid the negative quadrant.

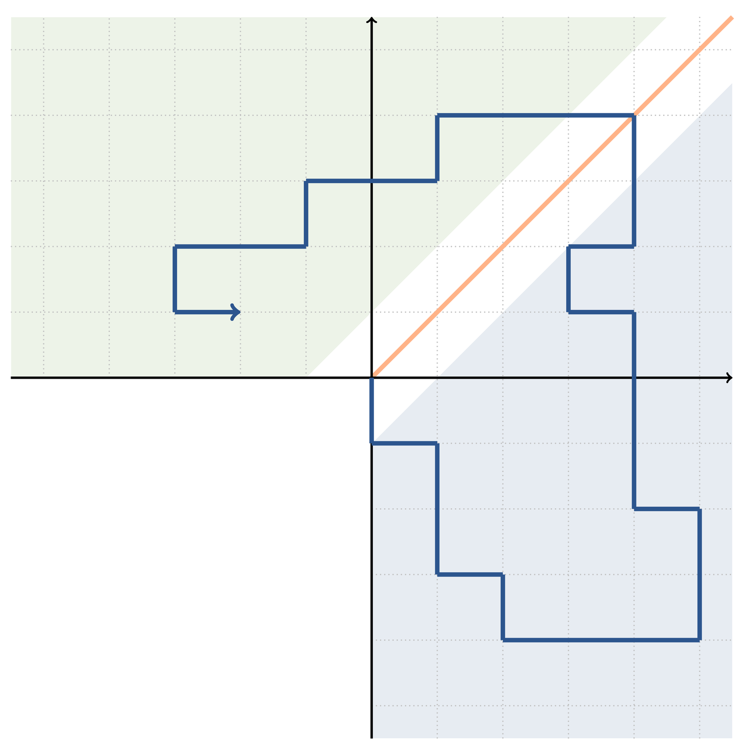

Figure 1: Simple walk in the three quarter plane.

Objective

The goal is to compute the number of paths $c(i,j;n)$ of length n, starting at $(0,0)$ and ending at (i,j), with ($i\geq 0$ or $j\geq 0$) and $n\geq 0$.

For example, $c(0,0;0)=1$ (the empty walk) and $c(0,0;2)=4$ ($\rightarrow\leftarrow$, $\leftarrow\rightarrow$, $\downarrow\uparrow$, and $\uparrow\downarrow$).
More generally, $c(0,0;n)=0$ for an odd n.

Method

A usual way to compute $c(i,j;n)$ is the following:

1. Consider the **generating function** of $c(i,j;n)$:
$$C(x,y)=\sum_{\substack{i\geq0 \text{ or } j\leq i\\ n\geq 0}} c(i,j;n)x^iy^jt^n;$$
2. Find a **functional equation** that $C(x,y)$ satisfies. For that, we decompose the domain of possible ends of the walk into three parts (see figure 1):
$$C(x,y)=L(x,y)+D(x,y)+L(y,x).$$
3. Solve the **functional equation**. Here, we use an analytic approach by transforming the functional equation into a **boundary value problem**.

Functional equation

We construct the walks step by step by concatenating a new step at each stage.

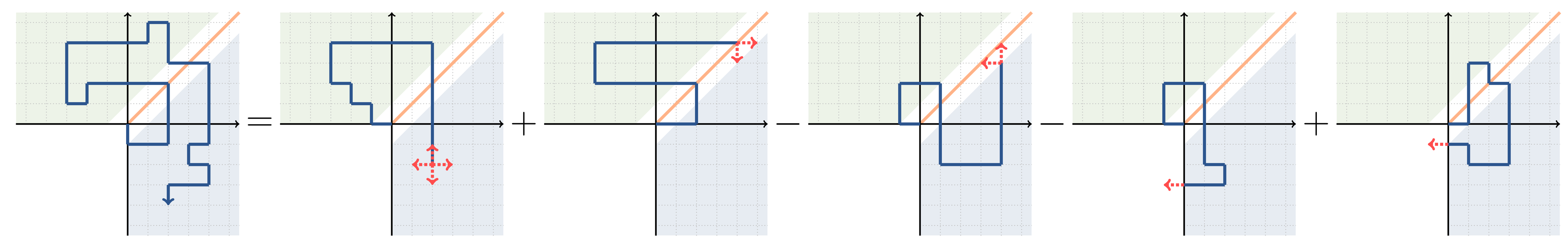

Figure 2: $L(x,y)=t(x+x^{-1}+y+y^{-1})L(x,y)+t(x+y^{-1})D(x,y)-t(x^{-1}+y)LD(x,y)-tx^{-1}L(0,y)+tx^{-1}\sum_{n\geq0}c(0,-1;n)y^{-1}t^n$.

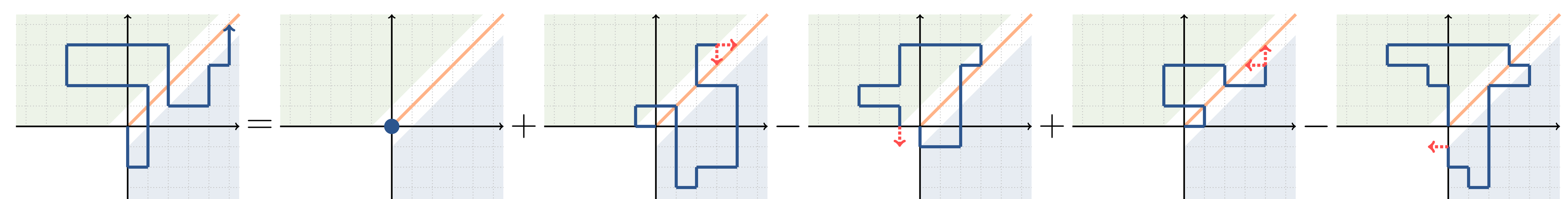

Figure 3: $D(x,y)=1+t(x+y^{-1})UD(x,y)-ty^{-1}\sum_{n\geq0}c(-1,0;n)x^{-1}t^n+t(x^{-1}+y)LD(x,y)-tx^{-1}\sum_{n\geq0}c(0,-1;n)y^{-1}t^n$.

Then, after a change of variable which simplifies the problem, we have:

Functional Equation

$$\tilde{L}(x,y)\widetilde{K}(x,y)xy=\frac{1}{2}xy-t\tilde{L}(x,0)+x\left(ty\,(xy+x)-\frac{1}{2}y\right)\widetilde{D}(y), \qquad (1)$$

with

$$\begin{cases}\tilde{L}(x,y) &= \sum_{\substack{i\geq1\\ j\leq0\\ n\geq0}} c(j,j-i;n)x^iy^jt^n,\\ \widetilde{D}(y) &= \sum_{\substack{i\geq0\\ n\geq0}} c(i,i;n)y^it^n,\\ \widetilde{K}(x,y) &= 1-t\,(x^{-1}+xy+x+x^{-1}y^{-1})\,.\end{cases}$$

Roots and branches of the Kernel

$\widetilde{K}(x,y)$ is called the **kernel** of the walks. We consider $\widetilde{X}(y)$ and $\widetilde{Y}(x)$, the two algebraic functions defined as $\widetilde{K}(\widetilde{X}(y),y)=0$ and $\widetilde{K}(x,\widetilde{Y}(x))=0$.

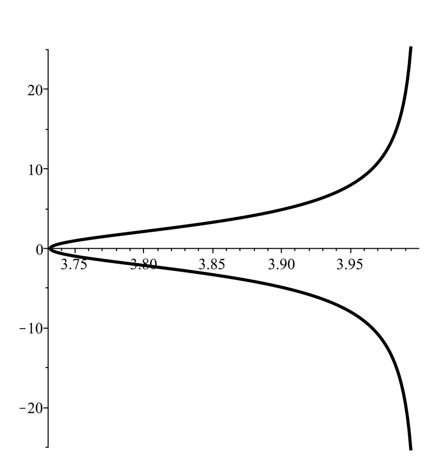

Figure 4: $\widetilde{X}([y_1,y_2])$.

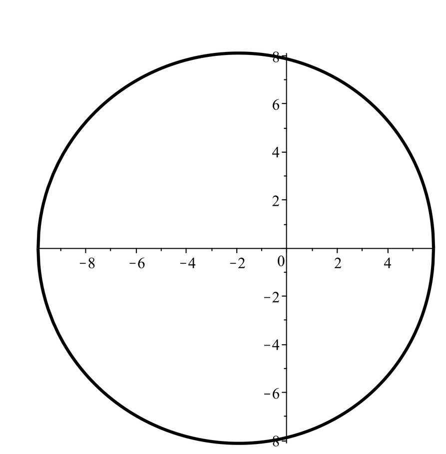

Figure 5: $\widetilde{Y}([x_1,x_2])$.

Boundary value problem

By evaluating equation (1) in $\widetilde{Y}$ and $\widetilde{X}$, we have the following **boundary value problem**:

Boundary value problem

For $y\in\widetilde{Y}([x_1;x_2])$,
$$R(y)\widetilde{D}(y)-R(\overline{y})\widetilde{D}(\overline{y})=y-\overline{y}. \qquad (2)$$
with
$$R(y)=y-2t\widetilde{X}_0(y)y(y+1).$$

Result

We solve the boundary value problem (2):

Contour integral expression

For y inside the curve $\widetilde{Y}([x_1,x_2])$,
$$\widetilde{D}(y)=\frac{\Psi(w(y))}{2i\pi}\times\int_{\widetilde{Y}_0([x_1,x_2])}\frac{(z-\bar{z})w'(z)dz}{R(z)\Psi^+(w(z))(w(z)-w(y))}, \qquad (3)$$
with:
$$\begin{cases}\Psi(y) &= (y-Y(x_1))\exp(\Gamma(y)),\\ \Psi^+(y) &= (y-Y(x_1))\exp(\Gamma^+(y)),\\ \Gamma(w(y)) &= \frac{1}{2i\pi}\int_{\widetilde{Y}_0([x_1,x_2])}\frac{\log(R(\bar{z})/R(z))w'(z)dz}{w(z)-w(y)}.\end{cases}$$
Γ^+ can be computed with the Sokhotski-Plemelj formulae,
w is a conformal gluing function.

Conclusion

Thanks to the expression of $\widetilde{D}$ in (3) and with the functional equation (1), we get $\tilde{L}(x,y)$. Then, thanks to the symmetry of the problem, we are able to find an **expression of** $C(x,y)$. To find $c(i,j;n)$, we still have to find an efficient way to extract the coefficient from the generating function. A further work could also be the study of the asymptotic of $c(i,j;n)$.

References

[1] K. Raschel.
Counting walks in a quadrant: a unified approach via boundary value problems.
J. Eur. Math. Soc. (JEMS), 14(3):749–777, 2012.

[2] G. Fayolle, R. Iasnogorodski, and V. Malyshev.
Random walks in the quarter plane.
Springer, Cham, second edition, 2017.

Acknowledgements

This is a joint work with Kilian Raschel. I thank both my supervisors Marni Mishna and Kilian Raschel for their advice and encouragements.